CAPACITY AND CHARACTERISTIC IMPEDANCE OF STRIP TRANSMISSION LINES WITH RECTANGULAR INNER CONDUCTORS[1]

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Abstract

An example of the rigorous solution of a complex region boundary value problem is presented. The particular problem, the electrostatic capacity and characteristic impedance of a strip transmission line, is solved exactly and numerical results are given for a particular geometrical configuration.

1. Introduction

A large class of important electromagnetic boundary value problems requiring the solution of Laplace's or Helmholtz' equation have boundaries which cannot be completely described by constant coordinate surfaces for the entire region under investigation. Consequently, the usual method of solving these problems by selecting a coordinate system which completely describes the boundaries of the region by constant coordinate surfaces cannot be used. Though this simple method of solution is inapplicable, there exists a systematic procedure for treating complex region boundary value problems, the so-called series-matching method.

The series-matching procedure of solving complex region boundary value problems can be divided into the following four steps:

1. The complex region for which the solution of Laplace's or Helmholtz's equations is desired is dissected into a number of simpler regions whose boundaries are constant coordinate surfaces. Eigenfunctions expansions of the fields in each of the simpler regions can then be selected such that they satisfy the boundary conditions peculiar to each of the simpler regions.

2. Equations relating the coefficients of the eigenfunction expansions in each of the simpler regions are obtained by imposing the continuity requirements of the fields across common boundaries of the simpler regions. Operations on these equations results in the writing of an infinite set of equations involving only the infinite set of eigenfunctions coefficients for one of the simpler regions.

3. The infinite set of equations are then solved with a demonstration given to show that the solution obtained is bounded and the required solution.

4. Finally, to minimize the labor involved in obtaining numerical results, such techniques as analytical continuation and the use of Kummer's transformation[2] are applied to simplify the solution obtained.

Examples of essentially the above procedure for the solution of electrostatic problems are given in the papers of Rigby and Knight[3,4]. Application of the above method to problems involving the solution of Helmholtz's equation have been given by Hahn, Whinnery, Jamieson and Robbins[5]. However, in these application to cavity and waveguide problems, the infinite set of equations obtained were not solved but were approximated by considering usually only the first four equations and coefficients of the infinite set and justifying the omission of the rest of the equations by the numerical smallness of the few calculated coefficients. Hahn's contribution to the series matching method, however, was in showing that a considerable simplification in the numerical handling of the infinite set of continuity equations results if some newly defined auxiliary functions are introduced. These auxiliary functions are used in Kummer transformations of terms appearing in the matching equations which are themselves slowly convergent series.

In this paper, a simple electrostatic problem will be solved using the series-matching method of solution so as to completely demonstrate the usefulness of this method of solution and to clearly illustrate the above mentioned solution procedure. Since the particular problem considered is useful in transmission line applications, the solution will be carried to numerical completeness giving the characteristic impedance of the configuration when used as a strip transmission line.

2. Formulation of Problem

Let us consider the calculation of the electrostatic capacity per unit length of an infinite length rectangular bar placed between two infinite extent ground planes. A plane cross-section perpendicular to the axis of the inner bar is shown in Fig. 1. The inner bar at potential $V = V_0$ and the infinite extent ground planes have infinite conductivity. The regions above, below, and to the right and left of the inner conductor have dielectric constants $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$, respectively.

The boundaries of the inner conductor can not be described by a single constant coordinate surface. Following step 1 of the procedure outlined above, the complex region between the inner bar and outer plates is dissected into four rectangular areas, where region (1) is bound by $|x| \leq \ell$, $a-d \leq y \leq a$, region (2) by $|x| \leq \ell$, $0 \leq y \leq b$, and region (3) by $|x| \geq \ell$, $0 \leq y \leq a$. The solution of Laplace's equation in the three regions satisfying the proper boundary conditions on the constant y planes peculiar to each region can now be immediately written.

2A. Solution of Laplace's Equation

The general solution of Laplace's equation for the potential V in regions (1), (2) and (3) satisfying the boundary conditions $V_1(x,a-d) = V_o$, $V_1(x,a) = 0$, $E_{1x}(x,a-d) = E_{1x}(x,a) = 0$ for $|x| \leq \ell$ in region (1), $V_2(x,b) = V_o$, $V_2(x,0) = 0$, $E_{2x}(x,b) = E_{2x}(x,0) = 0$ for $|x| \leq \ell$ in region (2) and $V_3(x,0) = V_3(x,a) = 0$, $E_{3x}(x,0) = E_{3x}(x,a) = 0$ for $|x| \geq \ell$ in region (3) is given by[6]

$$V_1(x,y) = \sum_{m=1}^{\infty} A_m \cosh\frac{m\pi x}{d} \sin\frac{m\pi}{d}(y-a+d) - \frac{V_o}{d}(y-a) \tag{1}$$

for $|x| \leq \ell$, $a-d \leq y \leq a$ in region (1),

$$V_2(x,y) = \sum_{n=1}^{\infty} B_n \cosh\frac{n\pi x}{b} \sin\frac{n\pi y}{b} + \frac{V_o}{b}\, y \tag{2}$$

for $|x| \leq \ell$, $0 \leq y \leq b$ in region (2), and

$$V_3(x,y) = \sum_{p=1}^{\infty} C_p\, e^{-p\pi|x|/a} \sin\frac{p\pi y}{a} \tag{3}$$

for $|x| \geq \ell$, $0 \leq y \leq a$ in region (3)[7].

2B. Continuity Equations

The triple set of coefficients A_m, B_n and C_p in (1), (2) and (3) can be determined by requiring the potential and displacement to be continuous across the common boundary at $|x| = \ell$ for the three regions. Using the orthogonal properties of the trigonometric functions, we obtain from the continuity of the potential from right to left at $|x| = \ell$

$$\frac{1}{2} C_q e^{-q\pi\gamma} = \left\{ \begin{array}{l} \dfrac{V_o}{(q\pi)^2}\left\{ \dfrac{\sin q\pi\alpha}{\alpha} - \dfrac{(-)^q \sin q\pi\beta}{\beta} \right\} \\ - \dfrac{\alpha}{\pi} \sin q\pi\alpha \displaystyle\sum_{n=1}^{\infty} \dfrac{(-)^n n B_n \cosh n\pi\gamma}{n^2 - q^2\,{}^2} \\ -(-)^q \dfrac{\beta}{\pi} \sin q\pi\beta \displaystyle\sum_{m=1}^{\infty} \dfrac{m A_m \cosh m\pi\delta}{m^2 - q^2\beta^2} \end{array} \right\} \tag{4}$$

where $\alpha = b/a$, $\beta = d/a$, $\gamma = \ell/b$ and $\delta = \ell/d$.[8]

Eq. (4) must be satisfied for all integer values of q; it therefore represents an infinite set of equations.

The continuity of the normal displacement D_x from left to right at $|x| = \ell$ for regions (1) and (3) gives

$$A_q = \frac{2}{\pi} \frac{\varepsilon_3}{\varepsilon_1} \beta \operatorname{csch} q\pi\delta \sum_{p=1}^{\infty} \frac{(-)^p\, p\, C_p\, e^{-p\pi\delta\beta} \sin p\pi\beta}{q^2 - p^2\beta^2} \tag{5}$$

Similarly, we obtain for the displacement continuity from region (2) to (3) at $|x| = \ell$

$$B_n = \frac{2(-)^n}{\pi} \frac{\varepsilon_3}{\varepsilon_2} \alpha \operatorname{csch} n\pi\gamma \sum_{p=1}^{\infty} \frac{p\, C_p\, e^{-p\pi\gamma\alpha} \sin p\pi\alpha}{n^2 - p^2\alpha^2}. \tag{6}$$

Substituting (5) and (6) into (4) for A_m and B_n, we obtain an infinite set of equations involving only the coefficients of region (3); namely,

$$c_m = a_m - \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} b_{mnp}\, c_p, \tag{7}$$

where

$$c_m = C_m e^{-m\pi\ell/a}, \tag{7a}$$

$$a_m = \frac{2V_o}{(m\pi)^2}\left\{ \frac{\sin m\pi\alpha}{\alpha} - \frac{(-)^m \sin m\pi\beta}{\beta} \right\}, \tag{7b}$$

and

$$b_{mnp} = \left\{ \begin{array}{l} \dfrac{4\varepsilon_3}{\pi^2}\left\{ \dfrac{\alpha^2 np \sin m\pi\alpha \sin p\pi\alpha \coth n\pi\gamma}{\varepsilon_2 (n^2 - m^2\alpha^2)(n^2 - p^2\alpha^2)} \right. \\ \left. + \dfrac{(-)^{m+p}\beta^2\, np \sin m\pi\beta \sin p\pi\beta \coth n\pi\delta}{\varepsilon_1 (n^2 - m^2\beta^2)(n^2 - p^2\beta^2)} \right\} \end{array} \right\} \tag{7c}$$

The solution of (7) and the use of (5) and (6), the potential, electrical field and displacement will be completely determined everywhere for the configuration shown in Fig. 1.

2C. Electrostatic Capacity

The electrostatic capacity C per unit length of the configuration shown in Fig. 1, immediately follows from the application of Gauss' theorem. Integrating the normal displacement on the outer plates or the inner conductor, we can write for the capacity per unit length

$$C = C_p + \Delta C, \tag{8}$$

where

$$C_p = 2\left\{ \varepsilon_1 \frac{\ell}{d} + \varepsilon_2 \frac{\ell}{b} \right\}, \tag{8a}$$

and

$$\Delta C = \frac{\pi\varepsilon_3}{V_o^2} \sum_{p=1}^{\infty} p\, a_p\, c_p. \tag{8b}$$

The a_p and c_p coefficients in (8b) are given by

(7b) and (7a), respectively. The C_p term in (8) is the per unit length parallel plate capacity that would exist between the inner and outer conductors if the fields in regions (1) and (2) were uniform. The ΔC term is therefore a correction to the parallel plate capacity required because of the distortion of the uniform field produced by the finite cross-section inner conductor and the penetration of the field into region (3).

2D. Characteristic Impedance

The configuration shown in Fig. 1 can be used as a strip transmission line. For the case of equal dielectric constants for the three regions ($\varepsilon_1 = \varepsilon_2 = \varepsilon_2 = \varepsilon$), a transverse electromagnetic wave (TEM) can be propagated whose characteristic impedance is given by

$$Z_k = \sqrt{\mu\, \varepsilon / C}\ , \qquad (9)$$

where μ is the permeability and C is the electrostatic capacity per unit length. Since for TEM wave propagation

$$LC = \mu \varepsilon\ , \qquad (10)$$

the inductance per unit length L of the transmission line is also determinable from C.

For the case of unequal dielectric constants as shown in Fig. 1., fulfillment of the continuity of the electric and magnetic fields at the interfaces of the different dielectric regions requires field components in the direction of propagation in addition to the transverse field components. The solution of Helmholtz's equation must be used therefore to derive the field components in the different regions instead of Laplace's equation as is done in the electrostatic and TEM case. However, the solution procedure for the electromagnetic field and Z_k for the unequal dielectric constants case parallels that given above for the electrostatic case, except that Z_k would be determined by use of Ohm's law or Poynting's vector theorem. Consequently, the expressions for C given below can be used in combination with (9) to determine Z_k only for the case of equal dielectric constants in the three regions.

3. Uniform Dielectric Symmetrical Case[9]

Equation (7) represents the culmination of steps (1) and (2) in the series-matching procedure of solving complex region potential problems. For the uniform dielectric ($\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon$) symmetrical case ($\alpha = \beta$), we can simplify (7) to read

$$c_m = a_m - \sum_{n=1}^{\infty} \sum_{\substack{p=1 \\ \text{odd}}}^{\infty} b_{mnp}\, c_p \qquad (11)$$

where

$$a_m = \frac{4V_o}{(\pi m)^2} \frac{\sin m\pi\alpha}{\alpha}\ , \qquad (11a)$$

and

$$b_{mnp} = 8\frac{\alpha^2}{\pi^2} \coth n\pi\gamma \frac{np \sin m\pi\alpha \sin p\pi\alpha}{(n^2 - m^2\alpha^2)(n^2 - p^2\alpha^2)}\ , \qquad (11b)$$

m and p odd only.

The a_m coefficients on the right side of (11) correspond to a linear potential distribution between the inner conductor and outer plates at $|x| = \ell$. Consequently, the double summation terms on the right side of (11) are the corrections to a linear potential required so that Laplace's equation and the boundary conditions are satisfied by the potential functions (1), (2), and (3).

3A. Solution of Series-Matching Equations

The particular form of (11) immediately suggests a solution by repeated substitution. Transforming the resultant repeated substitution solution so as to improve its rate of convergence, we obtain the following expression which is convergent for all values of α and γ and is the unique solution of (11):

$$c_m = \frac{4V_o}{\pi} \frac{\sin m\pi\alpha}{\pi\alpha} \left\{ \frac{1}{m^2} + 8\frac{\alpha^2}{\pi^2} \sum_{n=1}^{\infty} \frac{n}{\tanh n\pi\gamma + \frac{8}{\pi^2}\frac{U_n^o(\alpha)}{n}} \frac{\bar{H}_n(\alpha,\gamma)}{m^2\alpha^2 - n^2} \right\}, \qquad (12)$$

where

$$\bar{H}_n(\alpha,\gamma) = \sum_{r=0}^{\infty} \bar{h}_n^{(r)}(\alpha,\gamma)\ , \qquad (12a)$$

$$\bar{h}_n^{(o)} = -\frac{S_n^o(\alpha)}{n^2}\ , \qquad (12b)$$

and

$$\bar{h}_n^{(r+1)} = -\frac{8}{\pi^2} \sum_{q=1}^{\infty}{}' \frac{q}{\tanh q\pi\gamma + \frac{8}{\pi^2}\frac{U_q^o(\alpha)}{q}} \cdot \frac{S_n^o(\alpha) - S_q^o(\alpha)}{n^2 - q^2} \cdot \bar{h}_q^{(r)}. \qquad (12c)$$

In (12), m is an odd integer only; the prime in the summation in (12c) indicates that the q = n term is omitted in the sum.

The auxiliary functions introduced in (12) are

$$S_m^o(\alpha) = \sum_{\substack{r=1 \\ \text{odd}}}^{\infty} \frac{m^2 \sin^2 r\pi\alpha}{r(r^2\alpha^2 - m^2)} \quad (0 < \alpha < 1) \qquad (13)$$

and

$$U_m^o(\alpha) = n^2\alpha^2 \sum_{\substack{r=1\\ \text{odd}}}^{\infty} \frac{r \sin^2 r\pi\alpha}{(r^2\alpha^2 - n^2)^2} \quad (0 < \alpha < 1) \quad (14)$$

where the superscript o indicates only the odd terms are included in the summations. If the even terms were also included in the above sums, the series define the Hahn $S_m(\alpha)$ and $U_m(\alpha)$ functions.[5,10,11] The above functions can be determined from the Hahn functions by

$$W_m^o(\alpha) = W_m(\alpha) - \frac{1}{2} W_m(2\alpha) \quad , \qquad (15)$$

where W is either the S or U function.

Using (12) in combination with (5) and (6) and (1), (2), and (3), we can write the potential and fields throughout regions (1), (2) and (3) of Fig. 1. Letting $|x| = \ell$ in these expressions, the continuity of the potential and the displacement across the common $|x| = \ell$ boundary can be readily demonstrated.

3B. Capacity and Characteristic Impedance

Using (8) and (9), we have for the characteristic impedance of the uniform dielectric ($\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon$) symmetrical strip (b = d) transmission line in Fig. 1.

$$Z_k = \sqrt{\mu \ \varepsilon}/C \quad , \qquad (9)$$

$$C(\alpha,\gamma) = C_p + \Delta C = C_p + C_s + C_f \quad , \qquad (16)$$

where

$$C_p/\varepsilon = 4\ \ell/b \quad , \qquad (16a)$$

$$C_s/\varepsilon = \frac{16}{\pi^3} S_o^o(\alpha) \quad , \qquad (16b)$$

and

$$C_f/\varepsilon = \frac{128}{\pi^5} \sum_{n=1}^{\infty} \frac{\bar{H}_n(\alpha, \gamma)}{\tanh n\pi\gamma + \frac{8}{\pi^2}\frac{U_n^o(\alpha)}{n}} \cdot \frac{S_n^o(\alpha)}{n} \quad . \qquad (16c)$$

The function in (16b) is defined by the series

$$S_o^o(\alpha) = \sum_{\substack{r=1\\ \text{odd}}}^{\infty} \frac{\sin^2 r\pi\alpha}{\alpha^2 r^3} \qquad (17)$$

and is related to the Hahn $S_o(\alpha)$ by (15) The capacity of the strip line as indicated in (16) separates into three terms; the parallel plate capacity C_p, which would be produced by an uniform field extending only over the inner conductor for $|x| < \ell$, and two correction terms. The first correction C_s to the parallel plate capacity is a function of the inner conductor to outer plates spacing α and gives the additional stored energy contribution of region (3) to the capacity. The second correction term C_f which is a function of α and the normalized width γ of the inner conductor subtracts from the capacity $C_p + C_s$ and is the capacity correction required because of the uniform field distortion produced by the finite width and corners of the inner conductor.

4. Numerical Results

4A. Capacity and Characteristic Impedance

The tabulation of C_s/ε given in Table 1 for $0 \le \alpha(0.01) \le 0.50$ is based on the values of $S_o(\alpha)$ given in the Whinnery and Jamieson paper[12]. The form of C_f is particularly adaptable to automatic computing machine calculation because of the iterative process used in calcularing $\bar{H}_n(\alpha,\gamma)$. The normalized width of the inner conductor appears in (16c) and (12c) in the denominator as the argument of the tanh function. Consequently, the convergence of (16c) and (12a) will be the slowest for the case of $\gamma = \ell/b = 0$, the zero width strip. The capacity for the $\gamma = 0$ case, infinitely thin strip perpendicular to the outer plates, can be calculated by conformal mapping, however, and is given by

$$\frac{1}{\varepsilon} C(\alpha, 0) = \frac{4K'(k)}{K(k)} \quad , \qquad (18)$$

where K(k) is the complete elliptic integral of the first kind of modulus

$$k = \sin\pi a \qquad (18a)$$

and K' and k' are the complementary function and modulus, respectively[13]. Using (16), we can write

$$\frac{1}{\varepsilon} C_f(\alpha, 0) = \frac{1}{\varepsilon} C(\alpha, 0) - \frac{1}{\varepsilon} C_s(\alpha, 0) \ , \qquad (19)$$

where $C(\alpha, 0)$ and $C_s(\alpha, 0)$ are given by (18) and Table 1, respectively. From the numerical value given by (19), the number of times (12c) must be used (the upper limit in the r summation in (12a)) for a given accuracy in the value of C_f from (16c) can be determined. Using the same number of iterations for other values of γ, the numerical accuracy of C_f will be better than the upper bound determined in the $\gamma = 0$ case. For example, for $\alpha = 0.2$, we obtain from (19)

$$\frac{1}{\varepsilon} C_f(0.2, 0) = -0.321. \qquad (20)$$

The value of $\frac{1}{\varepsilon} C_f(0.2, 0)$ from (16c) for various values of the upper limit $\bar{r}$ in the summation in (12a) is as follows:

$\bar{r}$	$-\frac{1}{\varepsilon} C_f(0.2, 0)$
1	0.263
2	0.302
3	0.314
4	0.318

The accuracy of (16c) can also be checked for the case of $\gamma = \infty$, the infinitely wide strip, by using the conformal mapping solution[14]

$$\frac{1}{\varepsilon}\Delta C(\alpha,\infty) = \frac{4}{\pi}\left\{(\frac{1}{2\alpha}+1)\ \ln_e(\frac{1}{2\alpha}+1) - (\frac{1}{2\alpha}-1)\ \ln_e(\frac{1}{2\alpha}-1)\right\}. \quad (21)$$

Using (16), we can write

$$\frac{1}{\varepsilon}C_f(\alpha,\infty) = \frac{1}{\varepsilon}\Delta C(\alpha,\infty) - \frac{1}{\varepsilon}C_s(\alpha,\infty) \quad . \quad (22)$$

For the $\alpha = 0.2$ case again, we obtain from (22), (21), and Table 1

$$-\frac{1}{\varepsilon}C_f(0.2,\infty) = 0.137 \quad (23)$$

The value of $\frac{1}{\varepsilon}C_f(0.2,\infty)$ from (16c) for various values of the upper limit $\bar{r}$ in the summation in (12a) is as follows:

$\bar{r}$	$-\frac{1}{\varepsilon}C_f(0.2,\infty)$
3	0.136
4	0.137

Fig. 2 shows a plot of $-\frac{1}{\varepsilon}C_f(\alpha,\gamma)$ for the $\gamma = 0$ and $\gamma = \infty$ cases for $0 \leq \alpha \leq 0.5$. The maximum inaccuracy in the determination of the capacity and characteristic impedance contributed by $C_f(\alpha,\gamma)$ can be estimated from the above figure. The variation of $-\frac{1}{\varepsilon}C_f(0.2,\gamma)$ as a function of γ was determined by the use of (16c) and is shown in Fig. 3 and partially tabulated below in Table 2.[15] Table 2 shows the tabulation of the characteristic impedance for the $\alpha = 0.2$ case and various values of γ. For $\gamma \geq 1$ (width of strip equal to or greater than twice the inner to outer conductors spacing), the C_f term remains constant. Note, that for this particular value of α, omitting the $\frac{1}{\varepsilon}C_f$ term completely results in a maximum error of 6.5 per cent in the value of Z_k($\gamma = 0$ case).

4B. Electric Field at Corner of Inner Conductor

Differentiating either (2) or (3) and using (6) and (12), we obtain for the y component of the electric field at $|x| = \ell$ and $0 \leq y \leq b$

$$E_y(\ell,y) = -\frac{V_o}{b}\left(1 + \frac{8}{\pi^2}\sum_{m=1}^{\infty}\frac{(-)^m\, m\, \bar{H}_m(\alpha,\gamma)}{\tanh m\pi\gamma + \frac{8}{\pi^2}\frac{U_m^o(\alpha)}{m}}\cos\frac{m\pi y}{b}\right). \quad (24)$$

Letting $\alpha \to 0$ and $\gamma \to \infty$, we can use (24) to obtain the field at the corner across the gap between a right angle block and an infinite ground plane (see Fig. 4). This problem is, however, a simple conformal mapping problem and can be solved directly. W. R. Smythe has used the implicit expression for the gap field to write the following Fourier expansion:

$$E_y = -\frac{V_o}{b}\left(1 + \sum_{m=1}^{\infty} A_m \cos\frac{m\pi y}{b}\right), \quad (25)$$

where

$$A_m = (-)^m\left[0.299856\,\frac{\Gamma(m+0.1775)}{\Gamma(m+0.8225)} - \frac{0.031552}{m^2}\right] \quad (25a)$$

and $\Gamma(z)$ is the gamma function[16]. When $m = 1$, 0.006556 must be subtracted from (25a) to obtain the correct value for A_1. Using (24), we can write for $\alpha = 0$ and $\gamma = \infty$

$$A_m = \frac{8}{\pi^2}\ \frac{m(-)^m}{1 + \frac{8}{\pi^2}\frac{U_m^o\ 90)}{m}}\ \bar{H}_m(0,\infty) \quad . \quad (26)$$

Iterating (12c) four times to calculate the values of $\bar{H}_m$, the first twenty values of A_m determined by (25a) and (26) were compared. The difference between the two answers varied from one unit to three units in the fourth decimal as m increased to twenty.

5. Acknowledgement

The tabulation given for $C_f(0.2,\ )$ in 4A and the data presented in Fig. 3 were calculated by Mr. Donald R. Ryan.

References

1. This paper is a brief outline of a forthcoming Hughes Aircraft Company Technical Memorandum, TM-234A. Complete mathematical details of the results given here and the unsymmetrical transmission line calculations are presented in the Memorandum.

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6. M.K.S. units will be used throughout, where E and D are the electric field and displacement, respectively. The number subscript to V, E or D refers to the particular region of application for the expression.
7. Since the physical configuration in Fig. 1 is symmetrical about x = 0, only positive values of x need be considered.
8. Note, (4) is valid for α or β rational or irrational since for the rational case (α or β = r/s, where r and s are integers) the zero in the denominator of the n or m summations is removed in the limit by sine function zero.
9. The general case of the non-uniform dielectric and unsymmetrical spacing of the inner conductor is given in the Memorandum cited in reference 1.
10. C. Strachey and P. J. Wallis, "Hahn's Functions $S_m(\alpha)$ and $U_m(\alpha)$," Phil. Mag. (7), vol. 37, pp 87-94; February, 1946.
11. Extensive tabulation of (13) and (14) are given in the Memorandum cited in reference 1.
12. The value given for $S_o(0.05)$ is incorrect and should read 26.23879.
13. F. Oberhettinger und W. Magnus, "Anwendung der Elliptischen Funktionen in Physik und Technik," Springer-Verlag., Berlin, Germany, 1949, pp. 63 and 3.
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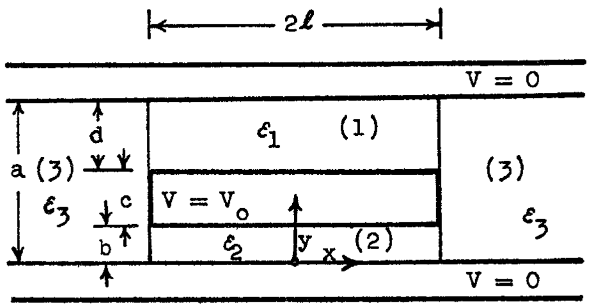


Fig. 1 - Generalize strip line configuration.

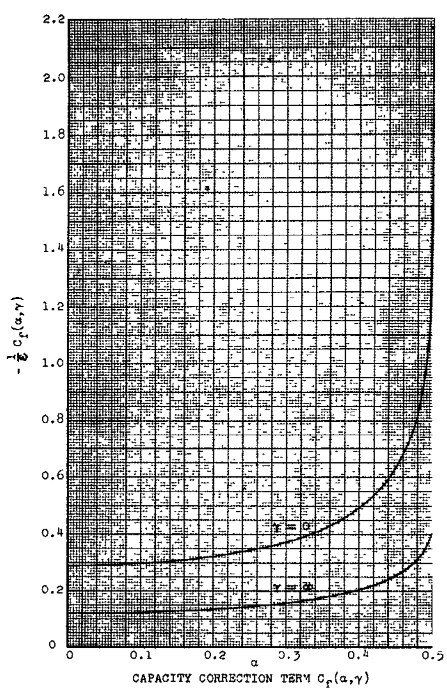


Fig. 2

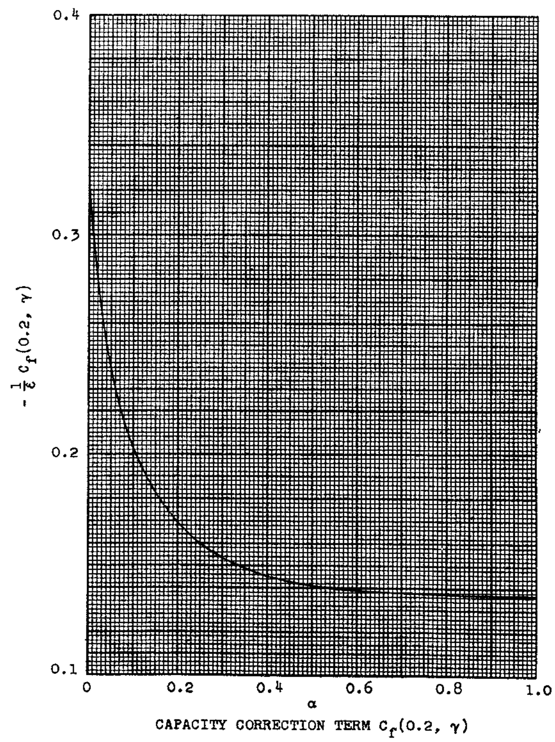


Fig. 3

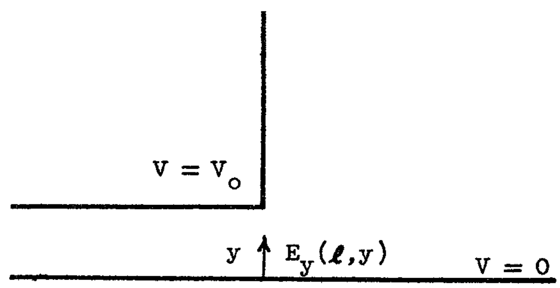


Fig. 4 - Corner gap.

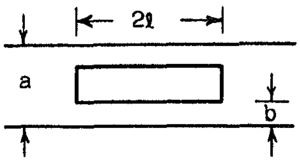
TABLE 1

Capacity Correction Term $\frac{1}{\varepsilon} C_s(\alpha)$

α	$\frac{1}{\varepsilon} C_s(\alpha)$	α	$\frac{1}{\varepsilon} C_s(\alpha)$
0.01	12.632	0.26	4.231
0.02	10.866	0.27	4.129
0.03	9.833	0.28	4.028
0.04	9.099	0.29	3.929
0.05	8.530	0.30	3.833
0.06	8.064	0.31	3.739
0.07	7.669	0.32	3.647
0.08	7.328	0.33	3.557
0.09	7.025	0.34	3.469
0.10	6.754	0.35	3.383
0.11	6.508	0.36	3.298
0.12	6.283	0.37	3.215
0.13	6.077	0.38	3.132
0.14	5.883	0.39	3.051
0.15	5.703	0.40	2.970
0.16	5.535	0.41	2.890
0.17	5.375	0.42	2.811
0.18	5.225	0.43	2.732
0.19	5.082	0.44	2.653
0.20	4.945	0.45	2.575
0.21	4.815	0.46	2.496
0.22	4.690	0.47	2.417
0.23	4.570	0.48	2.337
0.24	4.454	0.49	2.255
0.25	4.342	0.50	2.171

TABLE 2

Characteristic Impedance of a Strip Transmission Line



$\alpha = b/a = 0.2 \qquad \gamma = \ell/b$

γ	$\frac{1}{\varepsilon} C_p(\gamma)$	$\frac{1}{\varepsilon} C_s(0.2)$	$\frac{1}{\varepsilon} C_f(0.2,\gamma)$	$\frac{1}{\varepsilon} C(0.2,\gamma)$	$Z_k{}^*(0.2,\gamma)$
0	0	4.945	-0.321	4.624	81.53
0.05	0.20	4.945	-0.240	4.905	76.86
0.10	0.40	4.945	-0.203	5.142	73.32
0.20	0.80	4.945	-0.169	5.576	67.61
0.50	2.00	4.945	-0.141	6.804	55.41
1.00	4.00	4.945	-0.137	8.808	42.80
∞	∞	4.945	-0.137		

* $\sqrt{\mu/\varepsilon} = 120\pi$